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A HIGHER-ORDER BOUNDARY PERTURBATION METHOD FOR ASYMMETRIC DYNAMIC PROBLEMS IN SOLIDS—II. APPLICATION TO DISLOCATION RADIATION†

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Abstract—The boundary perturbation method developed in Part I is applied to investigate asymmetric radiation due to oscillating dislocations in solids. For an eccentrically located dislocation in a circular cylinder, the response is found to be in agreement with the exact solution. The response to dislocations oscillating about the centre of an elliptic cylinder is obtained and reveals a frequency spectrum markedly different from that of a circular cylinder.

1. INTRODUCTION

In Part I of this paper[1], general expressions for the application of the boundary perturbation method (BPM) have been derived for two classes of asymmetric problems: eccentric problems within a circular domain and elliptic problems. Making use of these expressions, we apply the BPM to two dynamic problems.

We consider first, in Section 2, the radiation due to a screw dislocation oscillating about an eccentric point of a circular cylinder. An exact solution to this problem was recently given by the authors[2]. The present BPM solution is seen to be much simpler and describes a behaviour which is in agreement with the exact results for moderate eccentricities.

In Section 3, we consider the radiation due to a screw dislocation oscillating about the centre of a cylinder having an elliptic cross-section. The exact solution to this problem would require a complex mathematical treatment involving Mathieu functions for which numerical evaluations of the resonance frequencies represents a considerable task. However, application of the BPM to this problem leads to a simple solution, providing readily the resonant frequencies which are of major importance, for example, in the interpretation of acoustic emission signals.

2. ECCENTRIC OSCILLATING SCREW DISLOCATION IN A CIRCULAR CYLINDER

2.1. *Formulation of the problem and the axisymmetric case*

We consider the radiation from a screw dislocation in an elastic cylinder of radius $r = a$ which oscillates radially with amplitude q and frequency ω about an equilibrium position, $x_0 = \eta a$, eccentric to the axis of symmetry (Figs 1 and 2). Hence, the coordinate position of the dislocation at any time t is

$$\xi_x(t) = x_0 + q\,e^{i\omega t}, \qquad \xi_y(t) = 0; \qquad |q|/a \leqslant \eta. \tag{2.1}$$

† Part of this work was performed by the first author at the Laboratoire de Mécanique des Solides, Ecole Polytechnique, Palaiseau, France, with the support of CNRS.

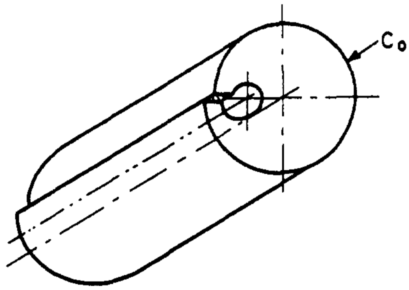

Fig. 1. Eccentric screw dislocation.

The oscillating screw dislocation may be described by a displacement field

$$u_r = u_\theta = 0, \qquad u_z = u(r, \theta)\,e^{i\omega t} \tag{2.2}$$

with $u(r, \theta)$ satisfying the equation of motion

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \theta^2} = -\chi^2 u, \tag{2.3}$$

where

$$\chi = \omega a / C_s \tag{2.4a}$$

represents a dimensionless wave number and

$$\rho = r/a, \tag{2.4b}$$

a dimensionless radial coordinate. In the above C_s is the wave speed of S-waves in the cylinder.

Eshelby has shown[3, 4] that for the displacement field defined by eqn (2.2), the oscillating dislocation may be prescribed by specifying the singularity

$$\lim_{\substack{\rho \to 0 \\ \theta \to 0}} u = \frac{bq}{2\pi a \rho} \sin \theta \tag{2.5}$$

on the displacement u, where b represents the Burgers vector. The remaining boundary

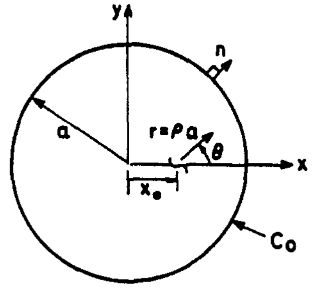

Fig. 2. Geometry of eccentric problem.

condition on the traction-free surface C_0 becomes, for the given displacement field,

$$\left.\frac{\partial u}{\partial n}\right|_{C_0} = 0, \tag{2.6}$$

where n is the normal to C_0.

The problem thus consists of finding a solution to the Helmholtz equation, eqn (2.3), subject to the boundary condition of eqn (2.6), and such that u satisfy the singularity given by eqn (2.5). We note, in passing, that eqn (2.3), being a homogeneous equation, the "forcing term" (as appears in eqn (I, 2.4a))† is instead represented by the singularity as $\rho \to 0$.

Now, proceeding with the BPM we let

$$u(\rho,\theta) = \sum_{j=0}^{3} \eta^j u^{(j)}(\rho,\theta). \tag{2.7}$$

Following the development of Sections 2 and 3 of Part I, substitution of eqn (2.7) in (2.3) leads to a set of homogeneous equations

$$\frac{\partial^2 u^{(j)}}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u^{(j)}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u^{(j)}}{\partial \theta^2} = -\chi^2 u^{(j)}; \qquad j = 0,1,2,3, \tag{2.8}$$

subject to the boundary conditions corresponding to eqn (2.6). In addition, we set

$$\lim_{\substack{\rho\to 0\\ \theta\to 0}} u^{(0)} = \frac{bq}{2\pi\rho a}\sin\theta, \tag{2.9}$$

that is, we let the $j = 0$ term satisfy the singularity. It follows then that for all $j \geqslant 1$, $u^{(j)}$ as $\rho \to 0$ must remain finite. Now, the solution for the $j = 0$ case, which corresponds to a dislocation oscillating about the axis of symmetry, has been given by Beltzer[5]; viz.

$$u^{(0)} = [A_1 J_1(\chi\rho) + B_1 Y_1(\chi\rho)]\sin\theta, \tag{2.10}$$

where

$$\begin{aligned} A_1 &= bq\chi Y_1'/4aJ_1', \\ B_1 &= bq\chi/4a. \end{aligned} \tag{2.11‡}$$

In the above J_n and Y_n are the Bessel functions of order n of the first and second kind, respectively. We observe that the $j = 0$ case is θ-dependent.

2.2. *Perturbed solutions*

For all cases $j \geqslant 1$, finite solutions of the Helmholtz equation, eqn (2.8), are of the form

$$u^{(j)}(\rho,\theta) = \sum_{n=1}^{\infty} \gamma_n^{(j)} J_n(\chi\rho)\sin n\theta. \tag{2.12}$$

The constants $\gamma_n^{(j)}$, evaluated from the boundary conditions corresponding to eqn (2.6), are

† References to equations appearing in Part I are indicated, e.g. by eqn (I, 2.4a), etc.

‡ For simplicity of notation, here and below, Bessel functions, appearing with no argument are evaluated at χ; i.e. $J_n \equiv J_n(\chi)$, etc. Furthermore, both here and below, derivatives of Bessel functions with respect to an argument are denoted by primes; e.g. $J_n' = \mathrm{d}J_n(\chi)/\mathrm{d}\chi$, $J_n'' = \mathrm{d}^2J_n(\chi)/\mathrm{d}\chi^2$, etc.

obtained for each case $j = 1, 2, 3$, by making use of the derived general expressions, eqn (I, 3.15).

We proceed to solve sequentially the set of problems $j = 1, 2, 3$. For the case $j = 1$, upon setting to zero the coefficient of the η term appearing in eqn (I, 3.15), the required condition becomes explicitly (with $f \equiv u$)

$$u_{,r}^{(1)}|_{\rho=1} = a \cos\theta\, u_{,rr}^{(0)}|_{\rho=1} + \frac{\sin\theta}{a} u_{,\theta}^{(0)}|_{\rho=1} \tag{2.13}$$

and upon substituting the solution for $u^{(0)}$, as given in eqn (2.10), we obtain

$$u_{,r}^{(1)}|_{\rho=1} = \frac{1}{2a}[A_1(\chi^2 J_1'' + J_1) + B(\chi^2 Y_1'' + Y_1)] \sin 2\theta. \tag{2.14}$$

Now, it is seen from eqn (2.12), that the boundary condition is satisfied if

$$\gamma_2^{(1)} = \frac{1}{2J_2'}\left[\chi(A_1 J_1'' + B_1 Y_1'') + \frac{1}{\chi}(A_1 J_1 + B_1 Y_1)\right],$$
$$\gamma_n^{(1)} = 0, \qquad n \neq 2. \tag{2.15}$$

For convenience we define the operator quantity

$$L_k \equiv \frac{d^k}{d\chi^k}[A_1 J_1(\chi) + B_1 Y_1(\chi)], \qquad k = 0, 2, 3, \ldots, \tag{2.16†}$$

where A_1 and B_1 are treated as constants. Hence,

$$u^{(1)} = \frac{1}{2J_2'}\left[\chi L_2 + \frac{1}{\chi} L_0\right] J_2(\chi\rho) \sin 2\theta. \tag{2.17}$$

Proceeding to the $j = 2$ problem, the general required boundary condition on $u^{(2)}$ at the boundary $\rho = 1$ is obtained by setting to zero the coefficient of the η^2 term appearing in eqn (I, 3.15).

Letting

$$Q(\chi) = \chi L_2 + \frac{1}{\chi} L_0, \tag{2.18a}$$

and noting that

$$\left.\frac{\partial^k u^{(1)}}{\partial r^k}\right|_{\substack{r=a\\(\rho=1)}} = \frac{Q(\chi)}{2J_2'(\chi)} (\chi/a)^k \frac{d^k J_2(\chi)}{d\chi^k} \sin 2\theta \tag{2.18b}$$

the boundary condition on $u^{(2)}$ can be written upon making appropriate substitutions, in the form

$$u_{,r}^{(2)}|_{\rho=1} = \frac{1}{2a}[M_1 \sin\theta + M_3 \sin 3\theta], \tag{2.19}$$

† Note that $L_1 = 0$ since eqn (2.11) satisfies the boundary condition for the $j = 0$ case; viz. $u_{,\rho}^{(0)}|_{\rho=1} = 0$.

where

$$M_1 = \frac{1}{4}(2L_0+3\chi^2L_2-\chi^3L_3)+\frac{Q(\chi)}{J_2'(\chi)}\left[\frac{\chi^2J_2''}{2}-J_2\right], \tag{2.20a}$$

$$M_3 = \frac{1}{4}(2L_0-\chi^2L_2-\chi^3L_3)+\frac{Q(\chi)}{J_2'}\left[\frac{\chi^2J_2''}{2}+J_2\right]. \tag{2.20b}$$

From the general solution,

$$u^{(2)} = \sum_{n=1}^{\infty}\gamma_n^{(2)}J_n(\chi\rho)\sin n\theta, \tag{2.21}$$

we observe, by matching like terms, that the boundary condition, eqn (2.19), is satisfied if

$$\begin{aligned}
&\text{(a)}\qquad \gamma_1^{(2)} = \frac{M_1}{2\chi J_1'},\\
&\text{(b)}\qquad \gamma_3^{(2)} = \frac{M_3}{2\chi J_3'},\\
&\text{(c)}\qquad \gamma_n^{(2)} = 0,\qquad n\neq 1 \text{ or } 3.
\end{aligned} \tag{2.22}$$

Hence $u^{(2)}$ becomes

$$u^{(2)} = \gamma_1^{(2)}J_1(\chi\rho)\sin\theta+\gamma_3^{(2)}J_3(\chi\rho)\sin 3\theta. \tag{2.23}$$

The $j = 3$ solution is obtained similarly upon setting to zero the coefficient of the η^3 term in eqn (I, 3.15) to obtain the explicit boundary condition and then comparing like terms. Omitting all tedious details of the algebraic manipulations, we present the results:

$$u^{(3)}(\chi,\rho) = \gamma_2^{(3)}J_2(\chi\rho)\sin 2\theta+\gamma_4^{(3)}J_4(\chi\rho)\sin 4\theta, \tag{2.24a}$$

where

$$\gamma_2^{(3)} = N_2/2\chi J_2'(\chi),\qquad \gamma_4^{(3)} = N_4/2\chi J_4'(\chi) \tag{2.25a, b}$$

with

$$N_2 = \frac{1}{4}(L_0-\chi^2L_2-\chi^3L_3+\chi^4L_4+\chi Q)+\frac{\chi^2Q(\chi)}{4J_2'}(J_2''-\chi J_2''')$$
$$+\gamma_1^{(2)}(J_1+\chi^2J_1'')-\gamma_3^{(2)}(3J_3-\chi^2J_3'') \tag{2.26a}$$

and

$$N_4 = \frac{1}{8}(L_0-\chi^2L_2+\chi^3L_3+\chi^4L_4/3-5\chi Q)-\frac{Q(\chi)}{4J_2'}(4J_2-\chi^2J_2''-\chi^3J_2''')$$
$$+\gamma_3^{(2)}(3J_3+\chi^2J_3''). \tag{2.26b}$$

Of particular importance is an identification of resonance frequencies. From eqns (2.10) and (2.11) we note that the $u^{(0)}$ term possesses resonances given by the roots of $J_1'(\chi) = 0$. Equation (2.17) reveals resonances in $u^{(1)}$ when $J_2'(\chi) = 0$, while eqns (2.21) and (2.22) and

(2.24) and (2.25) reveal additional resonances in $u^{(2)}$ and $u^{(3)}$ given by the roots of $J'_n(\chi) = 0$ for $n = 3$ and 4, respectively. Since the displacement is a linear combination of $u^{(j)}$, we observe that the third order BPM reveals resonances in u due to the roots of $J'_n(\chi) = 0$, $n = 1, 2, 3, 4$. It is reasonable to surmise therefore, that higher-order schemes would lead to resonant frequencies given by the roots of

$$J'_n(\chi) = 0, \qquad n = 1, 2, 3, \ldots, \infty. \tag{2.27}$$

Indeed this is precisely the results obtained from the exact solution, as given by the authors[2], where the resonant frequencies have been tabulated numerically. It is of interest to observe, however, that the BPM as developed here, leads to a much simpler solution requiring only a solution to a general homogeneous equation and the matching of simple boundary conditions. Moreover, the BPM solution presented here has the distinct capability of predicting the exact analytic pattern of resonant frequencies.

3. OSCILLATING SCREW DISLOCATION IN AN ELLIPTIC CYLINDER

3.1. *Formulation and perturbation of the problem*

We consider a screw dislocation oscillating with amplitude q and frequency ω along the semimajor axis and about the centre of an elastic elliptical cylinder having shear modulus μ and density ρ_D. The ellipticity, ε, of the cylinder is given as $\varepsilon = a/b - 1$ (Figs 3 and 4). The coordinate position of the dislocation thus, at any time t, is

$$\xi_x(t) = q\,e^{i\omega t}, \qquad \xi_y(t) = 0; \qquad |q| \ll a. \tag{3.1}$$

The basic governing equations of the present problem are essentially the same as those of the previous problem given in Section 2 (noting, however, that in accordance with the development in Part I, θ must be replaced by the coordinate ψ). The exception is the boundary condition of eqn (2.6) which must be replaced by

$$\left.\frac{\partial u}{\partial n}\right|_{C_e} = 0. \tag{3.2}$$

Consequently, the governing equations of the present problem are eqns (2.2)–(2.5) (with θ replaced by ψ) and eqn (3.2) above.

For the ellipse considered here, we let

$$u(\rho, \psi) = \sum_{j=0}^{2} \varepsilon^j u^{(j)}(\rho, \psi) \tag{3.3}$$

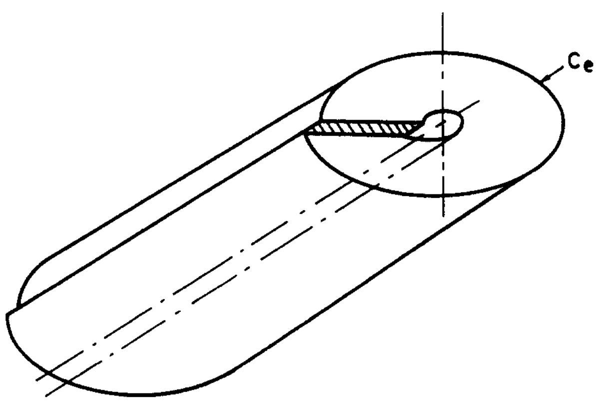

Fig. 3. Screw dislocation in elliptic cylinder.

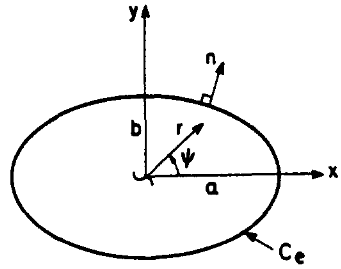

Fig. 4. Geometry of elliptic problem.

with $\rho = r/a$. Substituting in eqn (2.3), we obtain the set of homogeneous equations given by eqn (2.8) for $j = 0, 1, 2$ where as before $u^{(0)}$ behaves as $\rho \to 0$ according to the singularity of eqn (2.9) with $\theta \to \psi$. We note again that the $j = 0$ solution represents the case of the screw dislocation oscillating about the centre of a circular rod as given by eqns (2.10) and (2.11) with θ replaced by ψ.

Substituting eqn (3.3) in eqn (3.2) and using eqn (I, 4.20a) together with the defined quantities given in eqns (I, A.5), the boundary conditions on $u^{(j)}|_{C_0}$ for $j = 1$ and 2 become:

$$u^{(1)}_{,r}|_{r=a} = a \sin^2 \psi u^{(0)}_{,rr} - \frac{1}{a} u^{(0)}_{,\psi} \sin 2\psi|_{r=a}, \tag{3.4}$$

$$\begin{aligned} u^{(2)}_{,r}|_{r=a} = \Bigg[&\frac{1}{2}(\sin 2\psi)^2 u_{,r} - \frac{a}{2}(2 \sin^2 \psi - \cos^2 \psi) \sin^2 \psi\, u_{,rr} - \frac{a^2}{2} \sin^4 \psi\, u_{,rrr} \\ &+ \sin^2 \psi \sin 2\psi\, u_{,r\psi} - \frac{1}{4a} \sin 4\psi\, u_{,\psi} \Bigg]^{(0)} \\ &+ \left[a \sin^2 \psi\, u_{,rr} - \frac{1}{a} \sin 2\psi\, u_{,\psi} \right]^{(1)} \Bigg|_{r=a}, \end{aligned} \tag{3.5}$$

where $[\ldots]^{(j)}$ above denote the combination of derivatives of $u^{(j)}$.

Noting that the $j = 0$ solution satisfies the singularity at $\rho = 0$, finite solutions to the modified eqn (2.8) for $j > 0$ are

$$u^{(j)}(\rho, \psi) = \sum_{n=1}^{\infty} \bar{\gamma}^{(j)}_n J_n(\chi\rho) \sin n\psi, \tag{3.6}$$

where $\bar{\gamma}^{(j)}_n$ are constants to be evaluated by the boundary conditions, eqns (3.4) and (3.5). For the case $j = 1$, substitution of eqn (3.6) in eqn (3.4) leads, after considerable manipulation, and upon matching like terms in the resulting relation, to the following:

$$u^{(1)}(\rho, \psi) = \bar{\gamma}^{(1)}_1 J_1(\chi\rho) \sin \psi + \bar{\gamma}^{(1)}_3 J_3(\chi\rho) \sin 3\psi \tag{3.7}$$

with

$$\bar{\gamma}^{(1)}_1 = S_1/J'_1(\chi), \qquad \bar{\gamma}^{(1)}_3 = S_3/J'_3(\chi), \qquad \bar{\gamma}^{(1)}_n = 0, \qquad n \neq 1, 3. \tag{3.8}$$

In the above,

$$S_1 = (1/2)\left(\frac{3}{2}\chi L_2 - \frac{1}{\chi} L_0\right), \tag{3.9a}$$

$$S_3 = -(1/2)\left(\frac{\chi}{2}L_2 + \frac{L_0}{\chi}\right), \tag{3.9b}$$

where L_k is defined by eqn (2.16).

The second order perturbation $u^{(2)}$ may be obtained similarly. Substituting eqns (2.10) (with $\theta \to \psi$), (3.7) and (3.6), together with the defined quantities of eqns (3.8) and (3.9) in the boundary condition, eqn (3.5), taking the appropriate derivatives and making repeated use of the standard identity

$$\cos\alpha\psi\,\sin\beta\psi = (1/2)[\sin(\alpha+\beta)\psi + \sin(\beta-\alpha)\psi] \tag{3.10}$$

leads, after considerable manipulation, to the following result:

$$\bar{\gamma}_n^{(2)} = 0, \qquad n \neq 1, 3, 5, \tag{3.11}$$

from which,

$$u^{(2)} = \bar{\gamma}_1^{(2)} J_1(\chi\rho)\sin\psi + \bar{\gamma}_3^{(2)} J_3(\chi\rho)\sin 3\psi + \bar{\gamma}_5^{(2)} J_5(\chi\rho)\sin 5\psi. \tag{3.12}$$

The coefficients $\bar{\gamma}_n^{(2)}$ $(n = 1, 3, 5)$ are given by

$$\bar{\gamma}_1^{(2)} = \frac{1}{2J_1'}\left\{-\frac{9\chi}{8}L_2 - \frac{5\chi^2}{8}L_3 + \bar{\gamma}_1^{(1)}\left[\frac{3\chi}{2}J_1'' - \frac{J_1}{\chi}\right] + \bar{\gamma}_3^{(1)}\left[\frac{3J_3}{\chi} - \frac{\chi}{2}J_3''\right]\right\}, \tag{3.13a}$$

$$\bar{\gamma}_3^{(2)} = \frac{1}{2J_3'}\left\{-\frac{L_0}{4\chi} + \frac{11\chi}{16}L_2 + \frac{5\chi^2}{16}L_3 - \bar{\gamma}_1^{(1)}\left[\frac{J_1}{\chi} + \frac{\chi}{2}J_1''\right] + \bar{\gamma}_3^{(1)}\chi J_3''\right\}, \tag{3.13b}$$

$$\bar{\gamma}_5^{(2)} = -\frac{1}{4J_5'}\left\{\frac{L_0}{2\chi} + \frac{3\chi}{8}L_2 + \frac{\chi^2}{8}L_3 + \bar{\gamma}_3^{(1)}\left[\frac{6J_3}{\chi} + \chi J_3''\right]\right\}. \tag{3.13c}$$

Combining now the results for the $u^{(j)}$, we obtain from eqn (3.3)

$$\begin{aligned} u(\rho,\psi) = {} & [A_1 J_1(\chi\rho) + B_1 Y_1(\chi\rho)]\sin\psi + \varepsilon[\bar{\gamma}_1^{(1)} J_1(\chi\rho)\sin\psi + \bar{\gamma}_3^{(1)} J_3(\chi\rho)\sin 3\psi] \\ & + \varepsilon^2[\bar{\gamma}_1^{(2)} J_1(\chi\rho)\sin\psi + \bar{\gamma}_3^{(2)} J_3(\chi\rho)\sin 3\psi + \bar{\gamma}_5^{(2)} J_5(\chi\rho)\sin 5\psi]. \end{aligned} \tag{3.14}$$

3.2. *Numerical results and discussion*

According to the definitions of A_1, B_1, $\bar{\gamma}_n^{(1)}$, $\bar{\gamma}_n^{(2)}$ given above, we observe that for a screw dislocation oscillating in an elliptic cylinder, the second-order BPM reveals that resonances in the displacement u will occur at frequencies χ defined by the roots of

$$J_n'(\chi) = 0, \qquad n = 1, 3, 5. \tag{3.15}$$

We note additionally that the first-order scheme reveals resonances only for $n = 1$ and 3. On the other hand, we may surmise that using higher-order schemes, resonant frequencies χ will be found to occur for all odd n values satisfying $J_n'(\chi) = 0$. Thus, observing that for the case of a circular cylinder the sole resonances are given by the roots of $J_1'(\chi) = 0$, we may conclude that the effect of ellipticity is to introduce additional resonance for values $n = 3, 5, 7, \ldots$, thus radically changing the resonant response of the body.

For convenience, ordered resonance frequencies $\chi_{n,s}$ obtained from [6] are presented in Table 1 (where n and s represent the sth root of $J_n'(\chi) = 0$) for values $\chi \leqslant 10$.

It is noted that within this range of χ, three resonance frequencies, $\chi_{1,s}$ $(s = 1, 2, 3)$ exist in the case of a circular cylinder; the ellipticity gives rise to four additional frequencies.

In Figs 5 and 6 the normalized displacement, $u^* = ua/bq$, at points defined by $\psi = 45°$ and 90° on the boundary C_e is presented as a function of χ for elliptic cylinders with values

Table 1. Ordered resonant frequencies $\chi_{n,s}$, $0 < \chi < 10$

n	s	$\chi_{n,s}$	n	s	$\chi_{n,s}$
1	1	1.84118	3	2	8.01524
3	1	4.20119	1	3	8.53632
1	2	5.33144	7	1	8.57784
5	1	6.41562			

$\varepsilon = 0.2$ and 0.4, within the range $\chi \leqslant 8$. The response for the circular cylinder ($\varepsilon = 0$) is also presented for comparison. In Fig. 5, for the resonance response at a boundary point defined by $\psi = 45°$, we observe that the ellipticity has a radical effect upon the resonance response. For the resonance response for a boundary point at the semiminor axis, $\psi = 90°$, shown in Fig. 6, we observe the same resonances, although it is seen that the displacement pattern is considerably different from that of Fig. 5. A similar sensitivity to position was also noted in the case of the circular cylinder[2].

From Figs 5 and 6, we note that the curves are quite flat for low frequency values, $\chi < 1$, and therefore differ by little from values as $\chi \to 0$; hence, an approximate response for low frequencies can be obtained by taking the limit as $\chi \to 0$. Using the series representations of the Bessel functions, the quantities defined by eqns (2.11) and (2.16) are seen to tend to the following limits:

$$\begin{aligned}
&\lim_{\chi\to 0} A_1 bq/a = 1/\pi\chi, \qquad \lim_{\chi\to 0} B_1 bq/a = -\chi/4, \\
&\lim_{\chi\to 0} L_0 bq/a = 1/\pi, \qquad \lim_{\chi\to 0} L_2 bq/a = 1/\pi\chi^2, \qquad \lim_{\chi\to 0} L_3 bq/a = -3/\pi\chi^2.
\end{aligned} \tag{3.16}$$

Substituting in eqns (3.8), (3.9) and (3.13), the limits of $\bar{\gamma}_n^{(j)}$, $j = 1, 2$, are then readily obtained. Upon performing the detailed manipulations and combining eqn (3.14) in powers

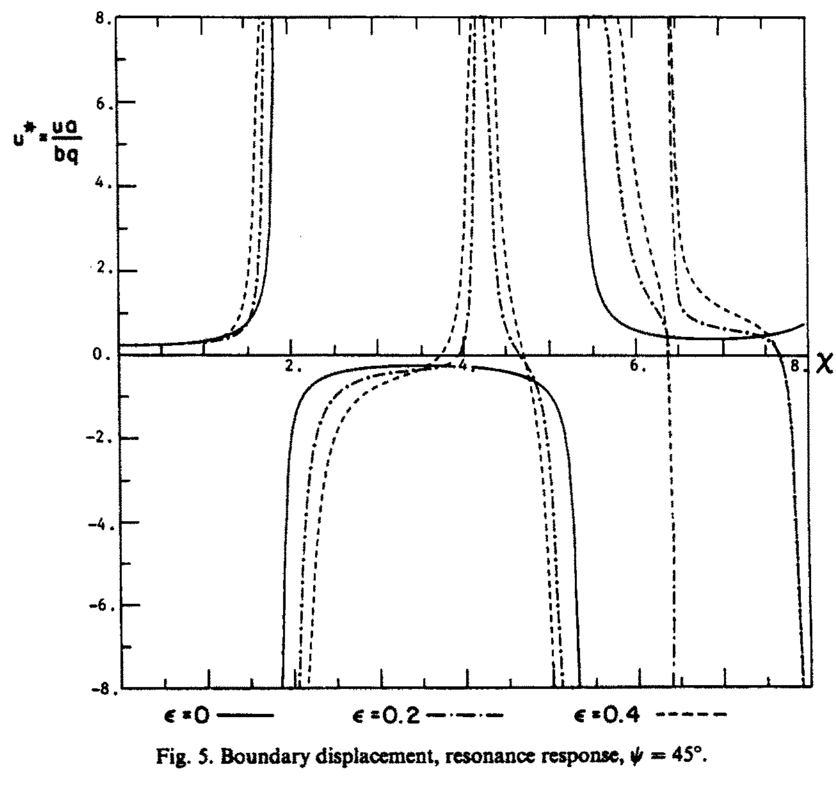

Fig. 5. Boundary displacement, resonance response, $\psi = 45°$.

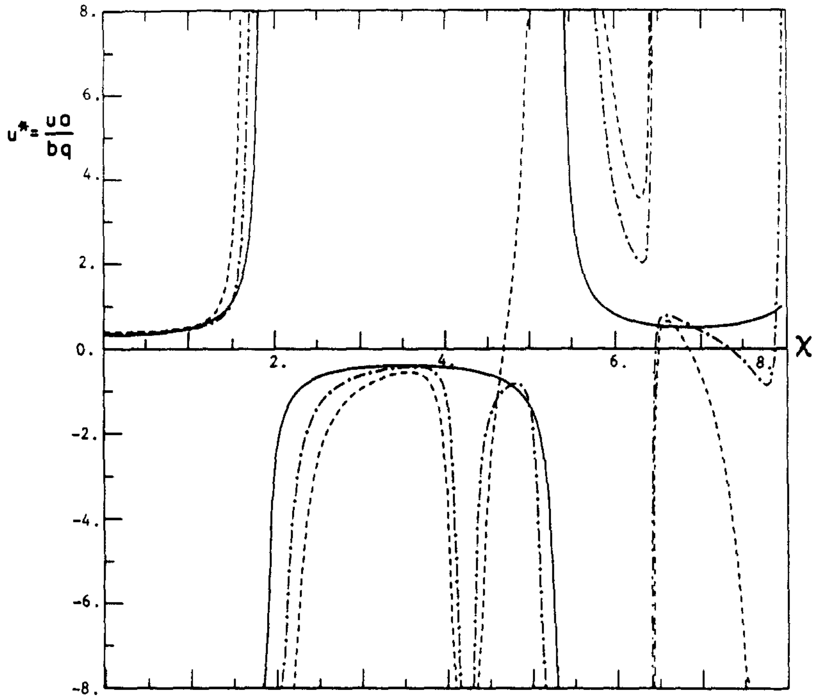

Fig. 6. Boundary displacement, resonance response, $\psi = 90°$.

of ε, the displacement $u(\rho, \psi)$ is observed to tend to the limit as follows:

$$\lim_{\chi\to 0}\frac{ua}{bq} = \frac{1}{2\pi}\left\{(1/\rho+\rho)\sin\psi + \frac{\rho}{2}(\sin\psi-\rho^2\sin 3\psi)\varepsilon + \frac{\rho}{2}\left(\sin\psi - \frac{3\rho^2}{2}\sin 3\psi + \frac{\rho^4}{2}\sin 5\psi\right)\varepsilon^2\right\} \quad (3.17)$$

from which we recover the singularity as $\rho \to 0$, given by eqn (2.9) with $\theta \to \psi$. Equation (3.17) thus represents the low frequency displacement field in an elliptic section resulting from a screw dislocation at the centre. The variation of the displacement along the boundary C_ε may then be obtained as a function of the coordinate ψ, from the derived expression for $\rho_\varepsilon = r/a|_{C_\varepsilon}$, eqns (I, 4.8) and (I, 4.9b). Substitution in eqn (3.17) leads, upon combining

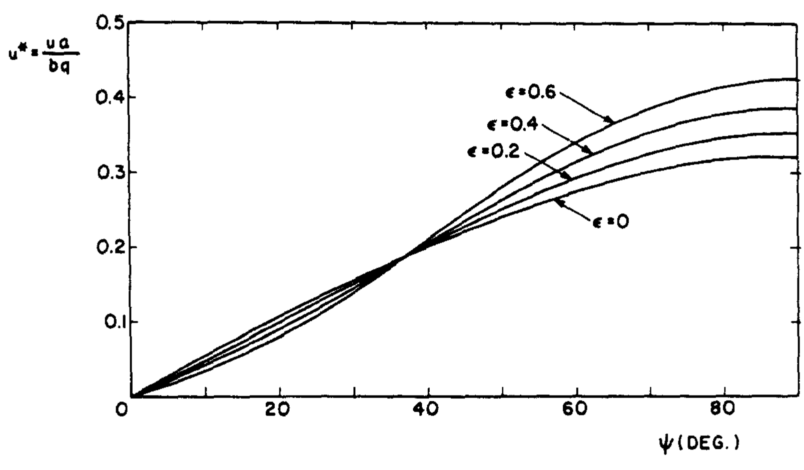

Fig. 7. Variation of low frequency boundary displacement, dependence on ellipticity.

terms appropriately, to the following:

$$\lim_{\alpha \to 0} \frac{ua}{bq}\bigg|_{C_\epsilon} = \frac{1}{2\pi}\left[2 \sin \psi + \frac{1}{2}(\sin \psi - \sin 3\psi)\varepsilon - \frac{\sin \psi}{256}(33 + 16 \cos 2\psi - 17 \cos 4\psi)\varepsilon^2\right]. \tag{3.18}$$

It is of interest to note that at $\psi = 90°$, the coefficient of ε^2 vanishes identically. The static variation of u on C_ϵ is shown in Fig. 7 for several values of the ellipticity ε in the range $0 \leqslant \psi \leqslant \pi/2$. (Values in the other quadrants follow from the symmetric/antisymmetric properties of the solution.)

As in the previous case, we observe that the BPM has led to a relatively simple treatment of a rather complex problem. It is clear that the exact solution to this problem would lead to expressions involving Mathieu functions for which numerical results, such as presented in Figs 5–7, would be difficult to obtain. The boundary perturbation method thus provides an alternative treatment which permits a direct identification of resonant frequencies and description of the behaviour of asymmetric systems.

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